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ON THE LOCATION OF ZEROS OF COMPLEX POLYNOMIALS

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ABSTRACT. This paper proves bounds for the zeros of complex valued polynomials. The assertions stated in this work have been specialized in the area of the location of zeros for complex polynomials in terms of two foci: (i) finding bounds for complex valued polynomials with special conditions for the coefficients and (ii) locating zeros of complex valued polynomials without special conditions for the coefficients $-$ especially we are searching for bounds, which again are positive roots of concomitant polynomials. As a result we obtain new zero bounds for univariate polynomials with complex coefficients.

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1. INTRODUCTION

The problems in the analytic theory of polynomials concerning locating zeros of complex polynomials have been frequently investigated. Over many decades, a large number of research papers, e.g, [1, 2, 3, 5, 6, 8, 9, 12, 13, 14] and monographs [7, 10, 11] have been published.

The new theorems in this paper provide closed disks in the complex plane

$$K(z_0, r) := \{z \in \mathbb{C} \,|\, |z - z_0| \leq r\}, \quad z_0 \in \mathbb{C}, \quad r \in \mathbb{R}_+,$$

containing all zeros of a complex valued polynomial. The steps to achieve this are as follows: Let

$$f_n : \mathbb{C} \longrightarrow \mathbb{C}; \quad f_n(z) = \sum_{i=0}^{n} a_i z^i, \quad a_i \in \mathbb{C}, \quad n \geq 1,$$

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be a complex polynomial. Construct a bound $\mathcal{S} = \mathcal{S}(a_0, a_1, \ldots, a_n)$ in such a way that all zeros of $f(z)$ are situated in the closed disk

$$K(z_0, \mathcal{S}(a_0, a_1, \ldots, a_n)) := \{z \in \mathbb{C} \mid |z - z_0| \leq \mathcal{S}(a_0, a_1, \ldots, a_n)\}.$$

Without loss of generality we set $z_0 = 0$. A special case is the existence of certain conditions, thus resulting in special bounds.

This paper is organized as follows: Section 2 provides such bounds on the basis of certain conditions, concerning the polynomials' coefficients. In Section 3 mainly zero bounds are proved which are positive roots of algebraic equations. The main result of Section 3 is a theorem which is an extension of a classical result of Cauchy. Section 4 shows applications of the bounds in such a way that the bounds will be evaluated with certain polynomials. The paper finishes in Section 5 with conclusions.

2. Bounds for the Zeros of Complex Polynomials with Special Conditions for the Coefficients

Definition 2.1. Let

$$f_n : \mathbb{C} \longrightarrow \mathbb{C}; \quad f_n(z) = \sum_{i=0}^{n} a_i z^i, \quad n \geq 1,$$

be a complex polynomial.

$$f_n(z) = a_n z^n + f_{n-1}(z), \quad f_0(z) := a_0 \tag{2.1}$$

denotes the recursive description of $f_n(z)$.

Theorem 2.1. *Let $P(z)$ be a complex polynomial, such that $P(z)$ is reducible in $\mathbb{C}[z]$,*

$$P(z) := f_{n_1}(z) g_{n_2}(z) = (b_{n_1} z^{n_1} + f_{n_1-1}(z))(c_{n_2} z^{n_2} + g_{n_2-1}(z))$$

with

$$|b_{n_1}| > |b_i|, \quad 0 \leq i \leq n_1 - 1, \quad |c_{n_2}| > |c_i|, \quad 0 \leq i \leq n_2 - 1.$$

If $n_1 + n_2 > 1$, then all zeros of the polynomial $P(z)$ lie in the closed disk

$$K(0, \delta), \tag{2.2}$$

where $\delta > 1$ is the positive root of the equation

$$z^{n_1+n_2+2} - 4z^{n_1+n_2+1} + 2z^{n_1+n_2} + z^{n_2+1} + z^{n_1+1} - 1 = 0. \tag{2.3}$$

It holds for $1 < \delta < 2 + \sqrt{2}$.

Proof. Expanding $f_{n_1-1}(z)$, we conclude

$$f_{n_1-1}(z) = \sum_{i=0}^{n_1-1} b_i z^i,$$

analogously

$$g_{n_2-1}(z) = \sum_{i=0}^{n_2-1} c_i z^i.$$

Assuming

$$|b_{n_1}| > |b_i|, \quad 0 \leq i \leq n_1 - 1, \quad |c_{n_2}| > |c_i|, \quad 0 \leq i \leq n_2 - 1, \tag{2.4}$$

it follows that

$$\begin{aligned}\frac{|f_{n_1-1}(z)|}{|b_{n_1}|} &= \frac{|\sum_{i=0}^{n_1-1} b_i z^i|}{|b_{n_1}|}\\ &\leq \sum_{i=0}^{n_1-1} \frac{|b_i|}{|b_{n_1}|}|z|^i\\ &< \sum_{i=0}^{n_1-1} |z|^i = \frac{|z|^{n_1}-1}{|z|-1}.\end{aligned}$$

With the inequalities (2.4) above, we find that

$$\begin{aligned}|P(z)| &= (b_{n_1} z^{n_1} + f_{n_1-1}(z))(c_{n_2} z^{n_2} + g_{n_2-1}(z))|\\ &\geq |b_{n_1}||c_{n_2}||z|^{n_1+n_2}\\ &\quad - \{|b_{n_1}||z^{n_1}||g_{n_2-1}(z)| + |c_{n_2}||z^{n_2}||f_{n_1-1}(z)| + |f_{n_1-1}(z)||g_{n_2-1}(z)|\}\\ &= |b_{n_1}||c_{n_2}|\left\{|z|^{n_1+n_2} - |\left[z|^{n_1}\frac{|g_{n_2-1}(z)|}{|c_{n_2}|} + |z|^{n_2}\frac{|f_{n_1-1}(z)|}{|b_{n_1}|}\right.\right.\\ &\quad\left.\left. + \ \frac{|f_{n_1-1}(z)|}{|b_{n_1}|}\cdot\frac{|g_{n_2-1}(z)|}{|c_{n_2}|}\right]\right\}\\ &> |b_{n_1}||c_{n_2}|\left\{|z|^{n_1+n_2} - \left[|z|^{n_1}\frac{|z|^{n_2}-1}{|z|-1} + |z|^{n_2}\frac{|z|^{n_1}-1}{|z|-1}\right.\right.\\ &\quad\left.\left. + \frac{(|z|^{n_1}-1)(|z|^{n_2}-1)}{(|z|-1)^2}\right]\right\}\\ &= \underbrace{\frac{|b_{n_1}||c_{n_2}|}{(|z|-1)^2}}_{>0}\{|z|^{n_1+n_2+2} - 4|z|^{n_1+n_2+1} + 2|z|^{n_1+n_2} + |z|^{n_2+1} + |z|^{n_1+1} - 1\},\end{aligned}$$

Let

$$H(|z|) := |z|^{n_1+n_2+2} - 4|z|^{n_1+n_2+1} + 2|z|^{n_1+n_2} + |z|^{n_2+1} + |z|^{n_1+1} - 1,$$

and we assume $n_1 + n_2 > 1$. Hence $|P(z)| > 0$ if $H(|z|) > 0$. Applying Descartes' Rule of Signs [4] to $H(|z|)$, we can conclude that $H(|z|)$ has either one or three positive zeros δ_1, δ_2, and δ_3.

Now we examine the positive zeros of $H(|z|)$. At first

$$H(1) = 0\,.$$

Dividing $H(|z|)$ by $|z| - 1$ yields

$$\begin{aligned}&(|z|^{n_1+n_2+2} - 4|z|^{n_1+n_2+1} + 2|z|^{n_1+n_2} + |z|^{n_2+1} + |z|^{n_1+1} - 1) : (|z| - 1)\\ &\qquad = |z|^{n_1+n_2+1} - 3|z|^{n_1+n_2} - \sum_{j=n_1+1}^{n_1+n_2-1} |z|^j + \sum_{j=0}^{n_2} |z|^j\,.\end{aligned}$$

Further, let

$$R(|z|) := |z|^{n_1+n_2+1} - 3|z|^{n_1+n_2} - \sum_{j=n_1+1}^{n_1+n_2-1} |z|^j + \sum_{j=0}^{n_2} |z|^j\,.$$

We see that $R(1) = 0$ and infer

$$H(|z|) = (|z| - 1)R(|z|) = (|z| - 1)^2 Q(|z|),\ \deg(Q(|z|)) = n_1 + n_2,$$

hence $\delta_1 = \delta_2 = 1$ is a zero with multiplicity two. Altogether, H has exactly three positive zeros. Now, we note that

$$\text{sign}\{H(0)\} = -1, \qquad \text{sign}\{H(\infty)\} = 1,$$

and we choose $K > 1$ such that $H(K) > 0$. On the other hand, it holds that $H(1) = H'(1) = 0$ and $H''(1) < 0$. Therefore, there exists a $\theta > 0$ such that $H(z) < 0,\ 1 \leq z \leq 1 + \theta$. Now, we obtain $H(1 + \theta) < 0 < H(K)$ and therefore $\delta_3 := \delta \in (1 + \theta, K)$, hence $\delta > 1$. Thus, $|P(z)| > 0$, if $H(|z|) > 0,\ |z| > \delta$. According to this, all zeros of $P(z)$ lie in the closed disk $K(0, \delta)$.

In order to examine the value of $\delta > 1$, we replace $|z|$ with δ in equation (2.3) and examine the equation

$$R(\delta) = \delta^{n_1+n_2+1} - 3\delta^{n_1+n_2} - \sum_{j=n_1+1}^{n_1+n_2-1} \delta^j + \sum_{j=0}^{n_2} \delta^j = 0.$$

The formulas

$$1 + \delta + \delta^2 + \cdots + \delta^{n_1} + \delta^{n_1+1} + \cdots + \delta^{n_1+n_2-1} = \frac{\delta^{n_1+n_2} - 1}{\delta - 1}$$

$$\iff \sum_{j=n_1+1}^{n_1+n_2-1} \delta^j = \frac{\delta^{n_1+n_2} - 1}{\delta - 1} - (1 + \delta + \delta^2 + \cdots + \delta^{n_1})$$

$$\iff \sum_{j=n_1+1}^{n_1+n_2-1} \delta^j = \frac{\delta^{n_1+n_2} - 1}{\delta - 1} - \frac{\delta^{n_1+1} - 1}{\delta - 1} = \frac{\delta^{n_1+n_2} - \delta^{n_1+1}}{\delta - 1}$$

yield the result

$$\delta^{n_1+n_2+1} - 3\delta^{n_1+n_2} - \frac{\delta^{n_1+n_2} - \delta^{n_1+1}}{\delta - 1} + \underbrace{\frac{\delta^{n_2+1} - 1}{\delta - 1}}_{>0} = 0\,.$$

From this equation follows the inequality

$$\delta^{n_1+n_2+1} - 3\delta^{n_1+n_2} - \frac{\delta^{n_1+n_2} - \delta^{n_1+1}}{\delta - 1} < 0,$$

and, furthermore,

$$\delta^{n_1+1}\left(\delta^{n_2+1} - 4\delta^{n_2} + 2\delta^{n_2-1} + 1\right) < 0\,. \tag{2.5}$$

Inequality (2.5) implies

$$\delta^{n_2}\left(\delta - 4 + \frac{2}{\delta}\right) < -1$$

and we finally conclude

$$\delta^2 - 4\delta + 2 < 0\,. \tag{2.6}$$

To solve inequality (2.6) take a closer look at

$$\xi^2 - 4\xi + 2 = 0,$$

and see that

$$\xi_{1,2} = 2 \pm \sqrt{2}\,.$$

With the fact that $\delta > 1$, the inequality holds for $1 < \delta < 2 + \sqrt{2}$. This completes the proof of Theorem 2.1. □

Theorem 2.1 can be strengthened by Theorem (2.2).

Theorem 2.2. *Let*

$$P(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0\,, \quad a_n \neq 0\,, \quad n \geq 1, \quad i = 0, 1, \ldots, n$$

be a complex polynomial such that $|a_n| > |a_i|,\ i = 0, 1, \ldots, n-1$. *Then all zeros of* $P(z)$ *lie in the closed disk* $K(0, 2)$.

Proof. For $|z| \leq 1$ the conclusion of Theorem 2.2 is evident. If we assume that $|z| > 1$ we obtain immediately

$$\begin{aligned} |P(z)| &= |a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0| \\ &\geq |a_n||z|^n - \{|a_{n-1}|z|^{n-1} + \cdots + a_0\} \\ &= |a_n||z|^n \left\{ 1 - \left[\frac{|a_{n-1}|}{|a_n|} \frac{1}{|z|} + \cdots + \frac{|a_0|}{|a_n|} \frac{1}{|z|^n} \right] \right\}. \end{aligned}$$

Now if $|a_n| > |a_i|,\ i = 0, 1, \ldots, n-1$ is assumed, we conclude

$$\begin{aligned} |P(z)| &\geq |a_n||z|^n \left\{ 1 - \left[\frac{|a_{n-1}|}{|a_n|} \frac{1}{|z|} + \cdots + \frac{|a_0|}{|a_n|} \frac{1}{|z|^n} \right] \right\} \\ &> |a_n||z|^n \left\{ 1 - \sum_{j=1}^{n} \frac{1}{|z|^j} \right\} \\ &> |a_n||z|^n \left\{ 1 - \sum_{j=1}^{\infty} \frac{1}{|z|^j} \right\} \\ &= |a_n||z|^n \left\{ 1 - \frac{1}{1 - \frac{1}{|z|}} + 1 \right\} \\ &= \underbrace{\frac{|a_n||z|^n}{|z| - 1}}_{>0} \{|z| - 2\}. \end{aligned}$$

Hence, we get

$$|P(z)| > 0 \quad \text{if} \quad |z| > 2.$$

□

3. FURTHER BOUNDS FOR THE ZEROS OF COMPLEX POLYNOMIALS

Now we prove Theorem 3.2, which is similar to Theorem 3.1. Theorem 3.1 is a classic result for the location of zeros found by Cauchy [7]. The zero bound of the new Theorem 3.2, as well as Theorem 3.1, depends on an algebraic equation's positive root.

Theorem 3.1 (Cauchy)**.** *Let*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0\,, \quad a_n \neq 0, \quad k = 0, 1, \ldots, n$$

be a complex polynomial. All zeros of $f(z)$ *lie in the closed disk* $K(0, \rho_C)$*, where* ρ_C *denotes the positive zero of*

$$H_C(z) := |a_0| + |a_1|z + \cdots + |a_{n-1}|z^{n-1} - |a_n|z^n.$$

Theorem 3.2. *Let*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0, \quad k = 0, 1, \ldots, n$$

be a complex polynomial. All zeros of $f(z)$ *lie in the closed disk* $K(0, \max(1, \delta))$*, where* $M := \max_{0 \le j \le n-1} \left| \frac{a_j}{a_n} \right|$ *and* $\delta \neq 1$ *denotes the positive root of the equation*

$$z^{n+1} - (1+M)z^n + M = 0.$$

Proof. On the basis of the inequality

$$|f(z)| \ge |a_n| \left[|z|^n - M\{|z|^{n-1} + \cdots + 1\} \right], \tag{3.1}$$

we get the modulus of $f(z)$

$$\begin{aligned} |f(z)| &\ge |a_n| \left[|z|^n - M\{|z|^{n-1} + \cdots + 1\} \right] \\ &= |a_n| \left[|z|^n - M \frac{|z|^n - 1}{|z| - 1} \right] \\ &= |a_n| \left[\frac{|z|^{n+1} - |z|^n(1+M) + M}{|z| - 1} \right]. \end{aligned}$$

Define

$$F(z) := z^{n+1} - z^n(1+M) + M.$$

Using the Descartes' Rule of Signs we have that $F(z)$ has exactly two positive zeros δ_1 and δ_2, and $F(\delta_1 = 1) = 0$ holds. With

$$\text{sign}\{F(0)\} = 1$$

and from the fact that $F(z)$ has exactly two positive zeros, we finally conclude that

$$|f(z)| > 0 \quad \text{for} \quad |z| > \max(1, \delta).$$

Hence, all zeros of $f(z)$ lie in $K(0, \max(1, \delta))$. □

Now, we express a further theorem, the proof of which is similar to that of Theorem 3.2.

Theorem 3.3. *Let*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0\,, \quad a_n \neq 0,\ k = 0, 1, \ldots, n$$

be a complex polynomial. All zeros of $f(z)$ *lie in the closed disk* $K(0, \max(1, \delta))$*, where*

$$\tilde{M} := \max_{0 \le j \le n} \left| \frac{a_{n-j} - a_{n-j-1}}{a_n} \right|, \quad a_{-1} := 0$$

and $\delta \neq 1$ *denotes the positive root of the equation*

$$z^{n+2} - (1+\tilde{M})z^{n+1} + \tilde{M} = 0.$$

Furthermore we can state immediately a new Theorem 3.4 which is very similar to a well known theorem due to Cauchy [7]. It provides an upper bound for all zeros of a complex polynomial $f(z)$. In many cases the bound of Theorem 3.4 is sharper than the bound of Cauchy [7] $(K(0, 1+M))$.

Theorem 3.4. *Let*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0\,, \quad a_n \neq 0, \quad k = 0, 1, \ldots, n$$

be a complex polynomial. All zeros of $f(z)$ *lie in the closed disk* $K(0, 1+\tilde{M})$.

Proof. For the zeros with $|z| \le 1$, we have nothing to prove. Assuming $|z| > 1$ and define

$$P(z) := (1-z)f(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z + a_0.$$

Now, we obtain the estimation

$$\begin{aligned}
|P(z)| &\ge |a_n||z|^{n+1} - \left\{|a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots + |a_1 - a_0||z| + |a_0|\right\} \\
&= |a_n|\Bigg\{|z|^{n+1} - \Bigg[\left|\frac{a_n - a_{n-1}}{a_n}\right||z|^n + \left|\frac{a_{n-1} - a_{n-2}}{a_n}\right||z|^{n-1} + \cdots \\
&\qquad + \left|\frac{a_1 - a_0}{a_n}\right||z| + \left|\frac{a_0 - 0}{a_n}\right|\Bigg\}\Bigg] \\
&\ge |a_n|\left\{|z|^{n+1} - \tilde{M}\sum_{i=0}^{n}|z|^i\right\} \\
&= |a_n|\left\{|z|^{n+1} - \tilde{M}\frac{|z|^{n+1} - 1}{|z| - 1}\right\} \\
&> |a_n|\left\{|z|^{n+1} - \tilde{M}\frac{|z|^n}{|z| - 1}\right\} \\
&= \underbrace{\frac{|a_n|}{|z| - 1}}_{>0}\left\{|z|^{n+2} - |z|^{n+1}(1 + \tilde{M})\right\}.
\end{aligned}$$

Finally, we conclude

$$|z|^{n+2} - |z|^{n+1}(1 + \tilde{M}) = |z|^{n+1}[|z| - (1 + \tilde{M})] > 0.$$

We infer $|P(z)| > 0$ if $|z| > 1 + \tilde{M}$. Hence, all zeros of $P(z)$ lie in $K(0, 1 + \tilde{M})$. Because of the fact that all zeros of $f(z)$ are zeros of $P(z)$, Theorem 3.4 holds also for $f(z)$. □

Now, we prove Theorem 3.5, which provides a zero bound for complex polynomials that are in a more general form than the previous ones.

Theorem 3.5. *Let*

$$f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0\,, \qquad a_n \ne 0, \qquad k = 0, 1, \ldots, n$$

be a complex polynomial and $H(z) := (f(z))^\sigma$, $\mathbb{N} \ni \sigma \ge 1$. *All zeros of* $H(z)$ *lie in the closed disk* $K(0, \rho)$, *where* ρ *is the positive zero of*

$$\text{(3.2)} \qquad \tilde{H}(z) := |a_n|^\sigma z^{n\sigma} - \left\{\left(\sum_{i=0}^{n-1}|a_i|z^i\right)^\sigma + \sum_{j=1}^{\sigma-1}\left[|a_n|^j z^{nj}\left(\sum_{i=0}^{n-1}|a_i|z^i\right)^{\sigma-j}\binom{\sigma}{j}\right]\right\}.$$

Proof. Define $H(z) := (f(z))^\sigma$, this yields the estimation

$$
\begin{aligned}
|H(z)| &= \left| \left(\sum_{i=0}^{n} a_i z^i \right)^\sigma \right| \\
&= \left| \left(\sum_{i=0}^{n-1} a_i z^i + a_n z^n \right)^\sigma \right| \\
&= \left| \sum_{j=0}^{\sigma} \left[(a_n z^n)^j \quad \left(\sum_{i=0}^{n-1} a_i z^i \right)^{\sigma-j} \binom{\sigma}{j} \right] \right| \\
&= \left| (a_n z^n)^\sigma + \sum_{j=0}^{\sigma-1} \left[(a_n z^n)^j \left(\sum_{i=0}^{n-1} a_i z^i \right)^{\sigma-j} \binom{\sigma}{j} \right] \right| \\
&\geq |a_n z^n|^\sigma - \left| \sum_{j=0}^{\sigma-1} \left[(a_n z^n)^j \left(\sum_{i=0}^{n-1} a_i z^i \right)^{\sigma-j} \binom{\sigma}{j} \right] \right| \\
&= |a_n z^n|^\sigma - \left| \left(\sum_{i=0}^{n-1} a_i z^i \right)^\sigma + \sum_{j=1}^{\sigma-1} \left[(a_n z^n)^j \left(\sum_{i=0}^{n-1} a_i z^i \right)^{\sigma-j} \binom{\sigma}{j} \right] \right| \\
&\geq |a_n z^n|^\sigma - \left\{ \left| \left(\sum_{i=0}^{n-1} a_i z^i \right)^\sigma \right| + \left| \sum_{j=1}^{\sigma-1} \left[(a_n z^n)^j \left(\sum_{i=0}^{n-1} a_i z^i \right)^{\sigma-j} \binom{\sigma}{j} \right] \right| \right\} \\
&\geq |a_n|^\sigma |z|^{n\sigma} - \left\{ \left(\sum_{i=0}^{n-1} |a_i||z|^i \right)^\sigma + \sum_{j=1}^{\sigma-1} \left[|a_n z^n|^j \left(\sum_{i=0}^{n-1} |a_i||z|^i \right)^{\sigma-j} \binom{\sigma}{j} \right] \right\} \\
&:= \tilde{H}(|z|)\,.
\end{aligned}
$$

$\tilde{H}(z)$ is a polynomial with only real and positive coefficients. Therefore we apply the Descartes' Rule of Signs and obtain that $\tilde{H}(z)$ has exactly one positive zero ρ.

Furthermore, we determine easily $\tilde{H}(0) = -|a_0|^\sigma < 0$. With

$$
\tilde{H}_1(z) := (a_n z^n)^\sigma \quad \text{and} \quad \tilde{H}_2(z) := \sum_{j=0}^{\sigma-1} \left[(a_n z^n)^j \left(\sum_{i=0}^{n-1} a_i z^i \right)^{\sigma-j} \binom{\sigma}{j} \right],
$$

we infer

$$
\deg(\tilde{H}_1(z)) > \deg(\tilde{H}_2(z)).
$$

Finally, it follows that all zeros of $H(z)$ lie in the closed disk $K(0, \rho)$. □

The following corollary states that the bound of Theorem 3.5 is a generalization of Cauchy's bound concerning Theorem 3.1.

Corollary 3.6. $\sigma = 1$ *in Equation (3.2) leads to the bound of Cauchy, Theorem 3.1.*

Proof. If we set $\sigma = 1$ in equation (3.2), we obtain

$$
\tilde{H}(z) = |a_n| z^n - \sum_{i=0}^{n-1} |a_i| z^i.
$$

Therefore, we have $\tilde{H}(z) = -H_C(z)$ and $\tilde{H}(z)$ posseses the same positive zero as $H_C(z)$. Thus we have the bound of Cauchy, Theorem 3.1. □

The last theorem in this paper expresses another bound for the zeros of $H(z)$. Here, the belonging concomitant polynomial is easier to solve that the concomitant polynomial of Theorem 3.5.

Theorem 3.7. *Let*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \,, \qquad a_n \neq 0, \qquad k = 0, 1, \dots, n$$

be a complex polynomial and $H(z) := (f(z))^\sigma$, $\mathbb{N} \ni \sigma \geq 1$. *Furthermore let* $B := \max_{0 \leq i \leq n-1} |a_i|$ *and* $K := \max_{0 \leq j \leq \sigma-1} |a_n|^j B^{\sigma-j} \binom{\sigma}{j}$. *All zeros of* $H(z)$ *lie in the closed disk* $K(0, \rho)$, *where* $\rho > 1$ *is the largest positive zero of the equation*

$$|a_n|^\sigma (|z| - 1)^\sigma - K|z|^\sigma - B^\sigma |z| + B^\sigma.$$

Proof. Starting from $\tilde{H}(|z|)$, obtained in the proof of Theorem 3.5 and the inequality

$$\sum_{i=0}^{n-1} a_i |z|^i \leq B \sum_{i=0}^{n-1} |z|^i,$$

we have

$$|H(z)| \geq |a_n|^\sigma |z|^{n\sigma} - \left\{ B^\sigma \left(\frac{|z|^n - 1}{|z| - 1} \right)^\sigma + \sum_{j=1}^{\sigma-1} \left[|a_n|^j |z|^{nj} B^{\sigma-j} \binom{\sigma}{j} \left(\frac{|z|^n - 1}{|z| - 1} \right)^{\sigma-j} \right] \right\}.$$

If we now assume $|z| > 1$ we obtain the inequalities

$$\begin{aligned} |H(z)| &> |a_n|^\sigma |z|^{n\sigma} - \left\{ B^\sigma \left(\frac{|z|^n - 1}{|z| - 1} \right)^\sigma + K \sum_{j=1}^{\sigma-1} |z|^{nj} \left(\frac{|z|^n - 1}{|z| - 1} \right)^{\sigma-j} \right\} \\ &> |a_n|^\sigma |z|^{n\sigma} - \left\{ B^\sigma \frac{|z|^{n\sigma}}{(|z| - 1)^\sigma} + K \sum_{j=1}^{\sigma-1} \frac{|z|^{n\sigma}}{(|z| - 1)^{\sigma-j}} \right\} \\ &= |z|^{n\sigma} \left\{ |a_n|^\sigma - \left[\frac{B^\sigma}{(|z| - 1)^\sigma} + \frac{K}{(|z| - 1)^\sigma} \sum_{j=1}^{\sigma-1} (|z| - 1)^j \right] \right\}. \end{aligned}$$

On the basis of the relation

$$\sum_{j=1}^{\sigma-1} (|z| - 1)^j < \sum_{j=0}^{\sigma-1} |z|^j,$$

we conclude furthermore that

$$\begin{aligned} |H(z)| &> |z|^{n\sigma} \left\{ |a_n|^\sigma - \left[\frac{B^\sigma}{(|z| - 1)^\sigma} + \frac{K}{(|z| - 1)^\sigma} \cdot \frac{|z|^\sigma - 1}{|z| - 1} \right] \right\} \\ &> |z|^{n\sigma} \left\{ |a_n|^\sigma - \left[\frac{B^\sigma}{(|z| - 1)^\sigma} + \frac{K|z|^\sigma}{(|z| - 1)^{\sigma+1}} \right] \right\} \\ &= \underbrace{\frac{|z|^{n\sigma}}{(|z| - 1)^{\sigma+1}}}_{>0} \left\{ |a_n|^\sigma (|z| - 1)^{\sigma+1} - K|z|^\sigma - B^\sigma |z| + B^\sigma \right\}. \end{aligned}$$

Define $P(z) := |a_n|^\sigma (z-1)^{\sigma+1} - Kz^\sigma - B^\sigma z + B^\sigma$. We see immediately that

$$\begin{aligned}\deg(P(z)) &= \deg\left(|a_n|^\sigma (z-1)^{\sigma+1} - Kz^\sigma - B^\sigma z + B^\sigma\right)\\ &= \deg\left(|a_n|^\sigma \sum_{j=0}^{\sigma+1}(-1)^j\binom{\sigma+1}{j} z^{\sigma+1-j} - Kz^\sigma - B^\sigma z + B^\sigma\right)\\ &= \sigma+1,\end{aligned}$$

hence we infer

$$\lim_{z\to\infty} P(z) = \infty. \tag{3.3}$$

Now, it holds

$$P(0) = \begin{cases} |a_n|^\sigma + B^\sigma & : \sigma \text{ is odd} \\ -|a_n|^\sigma + B^\sigma & : \sigma \text{ is even} \end{cases}$$

Furthermore we obtain $P(1) = -K < 0$. In the first case $P(0) = |a_n|^\sigma + B^\sigma > 0$ it follows that $P(z)$ has at least two positive zeros $\alpha_1 < \alpha_2$ and $\alpha_2 > 1$. The case $P(0) = -|a_n|^\sigma + B^\sigma > 0 \iff B^\sigma > |a_n|^\sigma$ leads us to the same situation. Now, we assume $P(0) = -|a_n|^\sigma + B^\sigma < 0 \iff B^\sigma < |a_n|^\sigma$ and it holds that $P(1) < 0$. With equation (3.3) we conclude that $P(z)$ has at least one positive zero $\alpha > 1$. Let $\alpha_1, \alpha_2, \dots, \alpha_\tau$, $1 \le \tau \le \deg(P(z))$ be the positive zeros of $P(z)$ and

$$\rho := \max_{1\le\tau\le\deg(\tilde{H}(x))} (\alpha_1, \alpha_2, \dots, \alpha_\tau).$$

Finally, we infer that $|P(z)| > 0$ if $|z| > \rho > 1$. This inequality completes the proof of Theorem 3.7. □

4. EVALUATION OF ZERO BOUNDS

Let $f(z) = \sum_{i=0}^n a_i z^i$, $a_n \neq 0$, $n \ge 1$, be a complex polynomial and $z_1, z_2, \dots, z_n$ be the zeros of $f(z)$. Define $\xi := \max(|z_1|, |z_2|, \dots, |z_n|)$ and

$$\begin{aligned} S_1 &:= \max(1,\delta), \text{ Theorem(2.1)}\\ S_2 &:= \delta, \text{ Theorem(3.2)}\\ S_3 &:= \delta, \text{ Theorem(3.3)}\\ S_4 &:= 1+\tilde{M}, \text{ Theorem(3.4)}\\ S_5 &:= \rho, \text{ Theorem(3.7)}\end{aligned}$$

Now, we consider the closed disks $|z| \le S_i$, $i = 1, 2, \dots, 4$ for each complex polynomial $f_j(z)$, $j = 1, 2, \dots, 6$.

(1)

$$\begin{aligned} f_1(z) &:= 360\cdot z^6 - 194z^5 - 79z^4 + 33z^3 + 11z^2 - 12z - 5\\ &= (18z^2 - 7z - 5)\cdot(20z^4 - 3z^3 + z + 1),\end{aligned}$$

therefore it holds

$$|b_2| > |b_i|, \quad 0 \le i \le 1, \quad |c_4| > |c_i|, \quad 0 \le i \le 3.$$

Zeros of $f_1(z)$:

$$\begin{aligned} z_1 &\doteq -0,3051 - 0,2775i\,, \\ z_2 &\doteq -0,3051 + 0,2775i\,, \\ z_3 &\doteq 0,3801 - 0,3864i\,, \\ z_4 &\doteq 0,3801 + 0,3864i\,, \\ z_5 &\doteq -0,3673\,, \\ z_6 &\doteq 0,7562\,. \end{aligned}$$

(2) $f_2(z) := z^5 - iz^4 + iz^2 - z + i$.
Zeros of $f_2(z)$:

$$\begin{aligned} z_1 &\doteq -1,1140 - 0,1223i\,, \\ z_2 &\doteq -0,8766i\,, \\ z_3 &\doteq 0,5950i\,, \\ z_4 &\doteq 1,5262i\,, \\ z_5 &\doteq 1,1140 - 0,1223i\,. \end{aligned}$$

(3) $f_3(z) := z^3 - 1000z^2 + 2000z + 1500$.
Zeros of $f_3(z)$:

$$\begin{aligned} z_1 &\doteq -0,5810\,, \\ z_2 &\doteq 2,5866\,, \\ z_3 &\doteq 997,9944\,. \end{aligned}$$

(4)

$$\begin{aligned} f_4(z) &:= 10z^4 + z^3 + (3+2i)z^2 + z(-2+i) + i \\ &= (5z^2 - 2z + i)\cdot(2z^2 + z + 1), \end{aligned}$$

therefore it holds

$$|b_2| > |b_i|, 0 \le i \le 1, |c_2| > |c_i|, 0 \le i \le 1.$$

Zeros of $f_4(z)$:

$$\begin{aligned} z_1 &= -0,25 - 0,6614i\,, \\ z_2 &= -0,25 + 0,6614i\,, \\ z_3 &= -0,1492 + 0,2863i\,, \\ z_4 &= -0,5492 + 0,2863i\,. \end{aligned}$$

(5) $f_5(z) := z^7 - z + 1$.

f_i	S_1	S_2	S_3	S_4	S_5	ξ	
f_1	3,2918	1,4895	2,5366	2,5388	2,1757 $(\sigma = 1)$	0,756	
f_2	-	1,9659	2,4069	2,4142	-	1,5262	
f_3	-	1501,0	3001,0	3001,0	-	997,9914	
f_4	3,1746	1,7897	1,8595	1,9	1.6683 $(\sigma = 1)$	1,25	
f_5	-	1,9919	1,2301	3,0	-	1,1127	
f_6	-	2,7494	4,2499	4,25	3,850 $(\sigma = 2)$	0,4184	

Figure 4.1: Comparison of zero bounds

Zeros of $f_5(z)$:

$$\begin{aligned} z_1 &\doteq -1,1127\,, \\ z_2 &\doteq -0,6170 - 0,9008i\,, \\ z_3 &\doteq -0,6170 + 0,9008i\,, \\ z_4 &\doteq 0,3636 - 0,9525i\,, \\ z_5 &\doteq 0,3636 + 0,9525i\,, \\ z_6 &\doteq 0,8098 - 0,2628i\,, \\ z_7 &\doteq 0,8098 + 0,2628i\,. \end{aligned}$$

(6) $f_6(z) := (2z^4 - z^3 + z^2 + 2z - 1)^2$, $\sigma = 2$.
Zeros of $f_6(z)$:

$$\begin{aligned} z_1 &\doteq -0,8935\,, \\ z_2 &\doteq 0,4184\,, \\ z_3 &\doteq 0,4875 - 1,0485i\,, \\ z_4 &\doteq 0,4875 + 1,04857i\,, \\ z_5 &\doteq -0,8935\,, \\ z_6 &\doteq 0,4184\,, \\ z_7 &\doteq 0,4875 - 1,04857i\,, \\ z_8 &\doteq 0,4875 + 1,04857i\,. \end{aligned}$$

5. CONCLUSIONS

Table 4.1 shows the overall results of the comparison of the new zero bounds. In general, the comparison between zero bounds is very difficult. Hence, it is not possible to obtain general assertions for describing the quality of zero bounds. Table 4.1 shows that the quality of the proven bounds depends on the polynomial under consideration. We observe that the bounds S_3, S_4 are very large for $f_3(z)$. This is due to the fact that the number $M := \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right|$ is very large. In the case of a polynomial $H(z) := (f(z))^\sigma$, the bound of Theorem 3.5 is directly applicable (without expanding $H(z)$) in order to determine the value of the bound. The new bounds of the present paper can be used in many applications. The characteristic property of our new bounds is that we can compute these zero bounds more effectively than the classic bound of Theorem 3.1.

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